An Interpretation of Airplane General Motion and Control as Inverse Problem

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A theoretical study for the inverse problem of the airplane general motion, that is, the problem of how an airplane should be controlled, when its flight maneuver is given, is described. An interpretation of this problem is made, and a general and practical method to solve the problem is developed. The distinctive features of this approach are to transform all the state variables of the airplane motion into the functions of the angle of attack α, sideslip angle β and bank angle φ, and to pay special attention to the distinction between the flight-path angles and the flight attitude angles. By treating the triplet (α, β, φ) as key variables it becomes easy to have insight into this complicated problem. This approach is applicable not only to conventional methods of motion and control but also to new and unusual methods through consideration of the degree of freedom of (α, β, φ) for a given maneuver. A numerical calculation of an interesting flight maneuver is presented and discussed to illustrate the problem.

Nomenclature

g	= gravitational acceleration
I_X, I_Y, I_Z, I_{ZX}	= moments of inertia and product of inertia
L, M, N	= components of total external moment
m	= airplane mass
P, Q, R	= angular velocity components
U, V, W	= velocity components
V_T	= tangent velocity
X, Y, Z	= components of total external force except gravity
x, y, z	= position components
X_u, Y_v, L_β, etc.	= dimensional stability derivatives
α, β, φ	= angle of attack, sideslip angle, bank angle
$\delta a, \delta e, \delta r, \delta T$	= control surfaces deflections from trim condition (aileron, elevator, rudder and power)
ψ, θ	= flight attitude angles (heading and pitch angles)
ψ_W, θ_W	= flight-path angles

Introduction

THIS paper is concerned with the inverse problem of airplane general motion, or the problem how an airplane should be controlled when its flight maneuver is given. Inverse manner itself is considered in the earlier reports by Weick and Jones,[1,2] and has been applied to several problems of maneuvering flight.[3] Particularly during the last ten years, the concept of the inverse problem has been effectively applied to the practical design of advanced flight control systems.[4-6] Because of the complexities and difficulties of mathematical treatment, the general theory of the problem has not been studied sufficiently. Reference 7 is a valuable source which deals systematically with the motion and control of guided bodies, but it is somewhat too general. The purpose of this paper is to interpret the motion and control of an airplane as inverse problem, i.e., to progress in a general and substantial way from the "motion" to the "control", not only for ordinary flights but also for complicated or unconventional flight maneuvers.

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In the ordinary problem, if the controls (primarily, three control surfaces and power) and the initial values of the state variables are given, the flight trajectory and the flight attitude can be determined in principle. In the inverse problem, however, if both flight trajectory (three degrees of freedom) and flight attitude (three degrees of freedom) are given, there will be no solution. If either flight trajectory or flight attitude alone is given, there will be many solutions in general. This is because the degree of freedom of the controlled motion for a conventional airplane is between three and four.

It seemed that the way of flying and the way of piloting have been settled. But, in the last ten years, new ways of flying have been observed with the spread of the CCV concept. A particular way of flying, whether conventional or not, depends on the control system. Therefore, consideration must be taken of these elements of the airplane when we treat the inverse problem generally.

The control surfaces of an airplane usually consist of the aileron, elevator, and rudder. They generate the moments about the X-, Y-, and Z-axes, respectively, and become the inputs of the moment equation of each axis. They are quite similar to each other except for acting on different axes. But they are not analogous in piloting the airplane. For example, the pitch angle is controlled by the elevator, but the yaw (heading) control is not performed, in the ordinary way, by the rudder. Because of the gap between the mathematical model and the piloting use of the control surfaces, the inverse problem can not be handled by the mathematical treatment alone.

The elevator, rudder, and aileron are regarded as the controls of the angle of attack α, the sideslip angle β, and the bank angle φ, respectively.[8] Therefore these angles (α, β, φ) are chosen as key variables, and an inverse problem is devoloped in a general way.

Equations

The equations of motion of an airplane in body axes are given by:

$$m\left\{\begin{bmatrix}\dot{U}\\ \dot{V}\\ \dot{W}\end{bmatrix}+\begin{bmatrix}0 & -R & Q\\ R & 0 & -P\\ -Q & P & 0\end{bmatrix}\begin{bmatrix}U\\ V\\ W\end{bmatrix}\right\} = mg\begin{bmatrix}-\sin\theta\\ \cos\theta\sin\varphi\\ \cos\theta\cos\varphi\end{bmatrix}+\begin{bmatrix}X\\ Y\\ Z\end{bmatrix} \tag{1}$$

$$\begin{bmatrix} I_X & 0 & -I_{ZX} \\ 0 & I_Y & 0 \\ -I_{ZX} & 0 & I_Z \end{bmatrix}\begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \times \begin{bmatrix} I_X & 0 & -I_{ZX} \\ 0 & I_Y & 0 \\ -I_{ZX} & 0 & I_Z \end{bmatrix}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \tag{2}$$

where the airplane is assumed to be a rigid body and has bilateral symmetry. The force equation [Eq. (1)] and the moment equation [Eq. (2)] are first-order differential equations with regard to velocity components (U,V,W) and angular velocity components (P,Q,R). The components (U,V,W) and (P,Q,R) are related to position component rates $(\dot{x},\dot{y},\dot{z})$ and Euler angle rates $(\dot{\psi},\dot{\theta},\dot{\varphi})$ as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi & \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{bmatrix}\begin{bmatrix} U \\ V \\ W \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & \sin\varphi/\cos\theta & \cos\varphi/\cos\theta \\ 0 & \cos\varphi & -\sin\varphi \\ 1 & \sin\varphi\tan\theta & \cos\varphi\tan\theta \end{bmatrix}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{4}$$

These kinematic equations are added to Eqs. (1) and (2) to make state equations. The components (X,Y,Z) and (L,M,N) shown in the right-hand side of Eqs. (1) and (2) denote the sum of external forces (except the gravity force) and moments acting on the airplane, respectively. They are mainly produced aerodynamically and are considered as the functions of $(U,V,W),(P,Q,R),(\delta a,\delta e,\delta r,\delta T)$ and their derivatives.

The state variables, which describe the motion of the airplane, are (U,V,W), (x,y,z), (P,Q,R), and (ψ,θ,φ). But using angle of attack α, sideslip angle β, and tangent velocity V_T, (U,V,W) can be written as

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = V_T \begin{bmatrix} \cos\beta\cos\alpha \\ \sin\beta \\ \cos\beta\sin\alpha \end{bmatrix} \tag{5}$$

Also using path angles (ψ_W,θ_W) and V_T, the position rates $(\dot{x},\dot{y},\dot{z})$ are written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = V_T \begin{bmatrix} \cos\theta_W\cos\psi_W \\ \cos\theta_W\sin\psi_W \\ -\sin\theta_W \end{bmatrix} \tag{6}$$

which is simpler than Eq. (3). The path angles should be distinguished from the attitude angles (ψ,θ). Note that the path angles (ψ_W,θ_W) are the two of the Euler angles of the wind axes $(\psi_W,\theta_W,\varphi_W)$ provided that the atmosphere is at rest. It also should be noted that φ_W and φ are not the same. For example, by the definition of the Euler angles (see next section), the relationship between φ and φ_W is

$$\tan\varphi_W = \frac{\sin\beta(\cos\alpha\sin\theta - \sin\alpha\cos\theta\cos\varphi) + \cos\beta\cos\theta\sin\varphi}{\sin\alpha\sin\theta + \cos\alpha\cos\theta\cos\varphi}$$

Attitude Angles and Path Angles

Let (ζ,η,ξ) be the Euler angles defined in flight dynamics, and $R(\zeta,\eta,\xi)$ be the rotation given by the Euler angles. Since the rotations can be expressed by means of orthogonal matrices, we have

$$R(\zeta,\eta,\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & \sin\xi \\ 0 & -\sin\xi & \cos\xi \end{bmatrix}\begin{bmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{bmatrix} \times \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Also, let (ψ,θ,φ) and $(\psi_W,\theta_W,\varphi_W)$ be the Euler angles for the body axes and wind axes, respectively. The body axes and wind axes are related by the angle of attack α and the sideslip angle β, and the rotation from the wind axes to the body axes can be shown by $R(-\beta,\alpha,0)$.[9] Then, the relation between both Euler angles is

$$R(-\beta,\alpha,0)\cdot R(\psi_W,\theta_W,\varphi_W) = R(\psi,\theta,\varphi) \tag{8}$$

From this equation the Euler angles of the body axes can be obtained in terms of the Euler angles of the wind axes and the aerodynamic angles α,β, and vice versa. For instance, the heading angle is shown below [see Eq. (10)]

$$\begin{aligned} \tan\psi &= \tan\psi(\psi_W,\theta_W,\varphi_W;\alpha,\beta) \\ &= [\cos\beta\cos\alpha\sin\psi_W\cos\theta_W - \sin\beta\cos\alpha(\sin\psi_W\sin\theta_W\sin\varphi_W \\ &\quad + \cos\psi_W\cos\varphi_W) - \sin\alpha(\sin\psi_W\sin\theta_W\cos\varphi_W - \cos\psi_W\sin\varphi_W)] \\ &\quad \div [\cos\beta\cos\alpha\cos\psi_W\cos\theta_W - \sin\beta\cos\alpha(\cos\psi_W\sin\theta_W\sin\varphi_W \\ &\quad - \sin\psi_W\cos\varphi_W) - \sin\alpha(\cos\psi_W\sin\theta_W\cos\varphi_W + \sin\psi_W\sin\varphi_W)] \end{aligned}$$

However, even though the flight path is given, the angle φ_W can not be determined. Further, the force equation [Eq. (1)] does not contain φ_W, but does contain φ. Therefore, what is required are the relations which do not contain φ_W, i.e.,

$$\psi = \psi(\psi_W,\theta_W;\alpha,\beta,\varphi) \tag{9a}$$

$$\theta = \theta(\psi_W,\theta_W;\alpha,\beta,\varphi) \tag{9b}$$

If so, it can be said that the heading and pitch are determined by the flight path and the triplet (α,β,φ).

For this purpose, multiplying Eq. (8) by $R^{-1}(-\beta,\alpha,0)$ from the left we have in the matrix form

$$\begin{bmatrix} \cos\psi_W\cos\theta_W & \sin\psi_W\cos\theta_W & -\sin\theta_W \\ \cos\psi_W\sin\theta_W\sin\varphi_W - \sin\psi_W\cos\varphi_W & \sin\psi_W\sin\theta_W\sin\varphi_W + \cos\psi_W\cos\varphi_W & \cos\theta_W\sin\varphi_W \\ \cos\psi_W\sin\theta_W\cos\varphi_W + \sin\psi_W\sin\varphi_W & \sin\psi_W\sin\theta_W\cos\varphi_W - \cos\psi_W\sin\varphi_W & \cos\theta_W\cos\varphi_W \end{bmatrix}$$
$$= \begin{bmatrix} \cos\beta\cos\alpha & \sin\beta & \cos\beta\sin\alpha \\ -\sin\beta\cos\alpha & \cos\beta & -\sin\beta\sin\alpha \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$
$$\times \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \cos\theta\sin\varphi \\ \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi & \cos\theta\cos\varphi \end{bmatrix} \tag{10}$$

From this, we obtain immediately

$$\cos\beta\cos\alpha\sin\theta - (\sin\beta\sin\varphi + \cos\beta\sin\alpha\cos\varphi)\cos\theta = \sin\theta_W \tag{11}$$

Noting the elements, which do not contain φ_W, of the left-hand side matrix in Eq. (10), and after some calculations, we also get

$$\sin(\psi_W - \psi) = \frac{\sin\beta\cos\varphi - \cos\beta\sin\alpha\sin\varphi}{\cos\theta_W} \tag{12}$$

When path angles (ψ_W, θ_W) and (α, β, φ) are given, the attitude angles (ψ, θ) can be calculated from Eqs. (11) and (12) without approximation.

General Approach

Although there are many ways to prescribe flight maneuvers, the most natural one is to give the flight path and the velocity on it. Therefore, an approach to this case is mainly considered. Other cases, for example, when the flight attitude is given, can be considered almost in the same way, and the details will be presented in the last section in this chapter.

Variables Transformation

To give the flight trajectory is to give the position which is the functions of time, $x(t), y(t), z(t)$. When they are given, by differentiating them with respect to time and solving Eq. (6) inversely, tangent velocity $V_T(t)$ and path angles $\psi_W(t)$, $\theta_W(t)$ are obtained. Or, more practically, $V_T(t), \psi_W(t), \theta_W(t)$ are given directly. If ψ_W and θ_W are known as the functions of time, then from Eqs. (11) and (12) we can get heading ψ and pitch θ as functions of time t and unknown variables $\alpha(t), \beta(t), \varphi(t)$, i.e.,

$$\psi = \psi(t, \alpha(t), \beta(t), \varphi(t)) \tag{13a}$$

$$\theta = \theta(t, \alpha(t), \beta(t), \varphi(t)) \tag{13b}$$

Angular velocities P, Q, R are expressed by $\theta, \varphi, \dot{\psi}, \dot{\theta}, \dot{\varphi}$ as shown in Eq. (4). Further, ψ, θ are expressed in Eqs. (13). Therefore, the angular velocities can be written in the form:

$$P = P(t, \alpha(t), \beta(t), \varphi(t), \dot{\alpha}(t), \dot{\beta}(t), \dot{\varphi}(t)) \tag{14a}$$

$$Q = Q(t, \alpha(t), \beta(t), \varphi(t), \dot{\alpha}(t), \dot{\beta}(t), \dot{\varphi}(t)) \tag{14b}$$

$$R = R(t, \alpha(t), \beta(t), \varphi(t), \dot{\alpha}(t), \dot{\beta}(t), \dot{\varphi}(t)) \tag{14c}$$

Velocities U, V, W are rewritten by V_T, α, β from Eq. (5). Since the trajectory has been given, $V_T(t)$ is known as the function of time. Accordingly, the velocities simply become the functions of unknown variables $\alpha(t)$ and $\beta(t)$:

$$U = U(t, \alpha(t), \beta(t)) \tag{15a}$$

$$V = V(t, \beta(t)) \tag{15b}$$

$$W = W(t, \alpha(t), \beta(t)) \tag{15c}$$

To sum up, by the trajectory $[x(t), y(t), z(t)]$ being given, all the remaining state variables (U, V, W), (P, Q, R), and (ψ, θ, φ) can be obtained as the functions which contain only α, β, φ as unknown variables.

Force Equations

The trajectory has been given. It is determined by the force equations what should first be controlled to fly the airplane along this trajectory. The second term of the right-hand side of Eq. (1), (X, Y, Z) are, as stated previously, considered as the functions of (U, V, W), (P, Q, R) and $(\delta a, \delta e, \delta r, \delta T)$. Therefore, if (X, Y, Z) is expressed in terms of these variables, then, by using Eqs. (13), (14) and (15), Eq. (1) will become differential equations which contain unknown variables (α, β, φ) and unknown controls $(\delta a, \delta e, \delta r, \delta T)$. In short, Eq. (1) can be written as follows:

$$F_X(t; \alpha, \beta, \varphi; \delta a, \delta e, \delta r, \delta T) = 0 \tag{16a}$$

$$F_Y(t; \alpha, \beta, \varphi; \delta a, \delta e, \delta r, \delta T) = 0 \tag{16b}$$

$$F_Z(t; \alpha, \beta, \varphi; \delta a, \delta e, \delta r, \delta T) = 0 \tag{16c}$$

where F may include also derivatives of the variables. If (α, β, φ) and $(\delta a, \delta e, \delta r, \delta T)$ satisfy Eqs. (16), the airplane can fly along the trajectory prescribed first. But, because α, β, φ and $\delta e, \delta r, \delta a$ are not independent, the moment equation [Eq. (2)] needs to be added which indicates this dependency. However, in practice, the control surfaces $(\delta a, \delta e, \delta r)$ are considered to play little role in the force equations, and only δT works in Eq. (16a). Accordingly, if the three controls in Eqs. (16) are neglected, then Eq. (2) need not be added, and Eqs. (16) are approximated as

$$F_X(t; \alpha, \beta, \varphi; \delta T) = 0 \tag{17a}$$

$$F_Y(t; \alpha, \beta, \varphi) = 0 \tag{17b}$$

$$F_Z(t; \alpha, \beta, \varphi) = 0 \tag{17c}$$

As a result, we can continue to discuss the problem by only these force equations for the present.

Since the above three simultaneous equations contain four unknown variables α, β, φ and δT, unique solutions are not likely to be obtained. That is, when only flight path is given, there is the degree of freedom of $[\alpha(t), \beta(t), \varphi(t)]$. This degree of freedom of (α, β, φ) becomes that of attitude angles (ψ, θ) from Eq. (9). If, from the first, attitude angles (ψ, θ, φ) are given arbitrarily in addition to flight path angles (ψ_W, θ_W), the aerodynamic angles (α, β) can be determined by solving Eq. (9) [the simultaneous equations of (α, β)]. But because of the arbitrariness these (α, β, φ) obtained do not satisfy Eq. (17) in general: the airplane can not fly along the flight path given first. In this respect, we can not give both flight path and flight attitude at the same time in a conventional sense.

Degree of Freedom of (α, β, φ)

To determine (α, β, φ) or flight attitude when flight trajectory is given, one more constraint equation must be added with regard to these variables. As previously mentioned, the control of α, β, φ is closely connected with the control system of the airplane. Therefore, the considerations on the control system and the way of piloting should be made before the triplet (α, β, φ) is determined. In the following, some cases will be considered as examples.

In ordinary cases the airplane is flown by maintaining sideslip angle or sideforce (the total force component along Y-axis except the gravity) zero, as is well known, which is a basis of piloting and called "coordinated". In this case, $\beta(t) \equiv 0$ is the additional constraint equation, and the remaining variables $\alpha(t), \varphi(t)$ and attitude angles $(\psi(t), \theta(t))$ can be determined uniquely (except inverted flights). This flight attitude is the most natural.

In the other cases, i.e., where the flight is not coordinated, two cases arise. In one case the flight is actively uncoordinated and in the other case passively. As an example of the former case, if the flight attitude is required to change to a certain degree from the natural one mentioned above, then, of course, $\beta(t) \equiv 0$ can not be satisfied, and (α, β, φ) must be determined so that they can meet this requirement (which is to be the additional condition). For practical examples, there are "aerobatics": in which the flight attitude is often unnatural for the flight path (e.g. see numerical example). Also there is a cross-wind landing method called "wing low" in which, in order to fly the airplane straight on the final approach with its heading ψ maintaining the runway azimuth, the sideslip angle β must be made owing to the existence of crosswind. But then, if the wings are level, the airplane begins to turn due to the β-sideforce and flies off the approach. Therefore, the bank angle φ also must be made in the direction of the crosswind.

The latter is the case where active control of β is not done. For examples there are "two-control turn"[8] in which elevator and either aileron or rudder are used to control α and φ, and the case in which rudder coordinations are not well performed. In these cases there are only two actual inputs in three moment equations. Therefore, even if (α, β, φ) are determined from the force equations by an arbitrary condition and substituted into the moment equations, there do not generally exist the two controls δe and δa (or, δe and δr) satisfying these equations. This means that the conditions determining (α, β, φ) are the moment equations themselves. Indeed, if the moment equations are added to the force equations, the number of equations agrees with that of unknown variables (see next section).

Another unconventional case is the one where an airplane has direct force control (DFC). Since direct-life (DL), δD_L and direct side-force (DSF), δD_{SF} enter the force equations explicitly unlike the conventional three controls, the force equations are changed essentially:

$$F_X(t; \alpha, \beta, \varphi; \delta T) = 0 \tag{18a}$$

$$F_Y(t; \alpha, \beta, \varphi; \delta D_{SF}) = 0 \tag{18b}$$

$$F_Z(t; \alpha, \beta, \varphi; \delta D_L) = 0 \tag{18c}$$

Thus the number of unknown variables becomes six, while the number of equations remains three. Accordingly, even though (α, β, φ) are prescribed quite arbitrarily, there still remain three variables δT, δD_{SF} and δD_L in Eqs. (18). Therefore, if δT, δD_{SF} and δD_L have no limitations of power, and also if α and β have no limitations of magnitude, the triplet (α, β, φ) can be chosen as we like, and so is the flight attitude. This is a well-known advantage of DFC.

Moment Equations

After the trajectory is given, the moment equations are also approximated as follows:

$$M_X(t; \alpha, \beta, \varphi; \delta a, \delta r) = 0 \tag{19a}$$

$$M_Y(t; \alpha, \beta, \varphi; \delta e) = 0 \tag{19b}$$

$$M_Z(t; \alpha, \beta, \varphi; \delta r, \delta a) = 0 \tag{19c}$$

In the previous sections, $(\alpha, \beta, \varphi; \delta T)$ were treated as inputs to the force equations, and $(\alpha, \beta, \varphi; \delta T)$ obtained by solving those equations were considered as the solutions of the inverse problem. Actually, the attitude angles can be obtained from (α, β, φ) by using Eqs. (11) and (12), and these (α, β, φ) are themselves not so inappropriate for the index of piloting. In these points the inverse problem was considered and solved without the moment equations. However, to get either deflections of control surfaces or more exact solutions, the moment equations are required. In the former case, by substituting $[\alpha(t), \beta(t), \varphi(t)]$ obtained into the moment equations and solving them with regard to the three controls, the control surface deflections $\delta a(t)$, $\delta e(t)$ and $\delta r(t)$ are obtained. In the latter case, the force and moment equations must be solved simultaneously. Although the number of equations and variables increase, the problem can be considered in the similar way.

Attitude Given Case

In general, it is impossible to prescribe $\varphi(t)$ rigidly together with $(\psi(t), \theta(t))$ for a conventional airplane. This is because the bank angle $\varphi(t)$ is not independent of the variations of the heading and pith angles $[\psi(t), \theta(t)]$. It should rather be considered that $\varphi(t)$ is one of the controls of $\psi(t)$ and $\theta(t)$, and is determined later. Therefore, only $\psi(t)$ and $\theta(t)$ are given at the start. This case is easier than the case of the flight path being given in the point that $\psi(t)$ and $\theta(t)$ are given directly as the functions of time [see Eq. (13)].

In a way similar to Eq. (17), the force equation [Eq. (1)] become

$$F_X(t; V_T, \alpha, \beta, \varphi; \delta T) = 0 \tag{20a}$$

$$F_Y(t; V_T, \alpha, \beta, \varphi) = 0 \tag{20b}$$

$$F_Z(t; V_T, \alpha, \beta, \varphi) = 0 \tag{20c}$$

Thus, the number of unknown variables becomes five. By the same argument made before, two more constraint equations can be added to get unique solutions. Therefore, it seems here that $\varphi(t)$ can also be prescribed. But, if this is done, it will affect the solutions of the other variables, and they may become unrealizable in effect, e.g., the magnitude of the sideslip angle $\beta(t)$ may be extremely large in order to produce the sideforce, or the lift may be inverted.

In usual cases, the tangent velocity $V_T(t)$ and the condition of the coodination $\beta(t) \equiv 0$ are given here. Then the force equations [Eq. (20)] are

$$F_X(t; \alpha, \varphi; \delta T) = 0 \tag{21a}$$

$$F_Y(t; \alpha, \varphi) = 0 \tag{21b}$$

$$F_Z(t; \alpha, \varphi) = 0 \tag{21c}$$

By solving the eqs. (21) simultaneously the angle of attack $\alpha(t)$, the bank angle $\varphi(t)$, and the power $\delta T(t)$ are obtained as the solutions of this inverse problem. The succeeding procedures and the considerations about unusual cases are the same as before. The path angles $\theta_W(t)$ and $\psi_W(t)$ can be obtained from Eq. (11) and the following equation [from Eq. (10)]:

$$\begin{aligned}\tan\psi_W &= \tan\psi_W(\psi, \theta, \varphi; \alpha, \beta)\\ &= [\cos\beta\cos\alpha\sin\psi\cos\theta + \sin\beta(\sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi)\\ &\quad + \cos\beta\sin\alpha(\sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi)]\\ &\quad \div [\cos\beta\cos\alpha\cos\psi\cos\theta + \sin\beta(\cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi)\\ &\quad + \cos\beta\sin\alpha(\cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi)]\end{aligned} \tag{22}$$

In the present approach, one of the methods of considering the problem is counting the numbers of equations and unknown variables, which is well-known. It may seem that this is not strict from the mathematical viewpoint, but because of the nonlinearity of the simultaneous equations, there is nothing else that is of practical use for examining the existence and uniqueness of the solution. Therefore, physical conditions should be taken into account, together with this method, for each individual case where the form of the equation is clearly decided.

Numerical Example

Basic Solution

An interesting flight maneuver is now considered, and a numerical example of the inverse problem is illustrated. That is, the triplet (α, β, φ), flight attitude, and the controls of this flight are calculated by the procedure shown in the preceding section. Suppose one maneuver in which an airplane is flown along a straight and level flight path with a 360-deg continuous rolling. Such a maneuver looks like actual aerobatics, what is called aileron rolls or slow rolls, but it is an imaginary one in the point that it is supposed that the flight path must be exactly straight. Therefore, it is expected that the control of this maneuver differs from that of those maneuvers. The flight attitude and the three controls of this maneuver depend on the variations of α and β taken with that of φ along the trajectory.

First of all, the flight trajectory is given as follows

$$[V_T(t), \psi_W(t), \theta_W(t)] \equiv (V_T, 0, 0), \qquad V_T = \text{Const.} \quad (23)$$

Since this is a nonaccelerated flight, the acceleration terms, i.e., the left-hand sides of Eq. (1) become zero. Thus, assuming linear aerodynamic forces and using dimensional stability derivatives, Eq. (1) become

$$-g\sin\theta + X_u u + X_w w + X_{\delta T}\delta T = 0 \quad (24a)$$

$$g\cos\theta\sin\varphi + Y_v v = 0 \quad (24b)$$

$$g(\cos\theta\cos\varphi - 1) + Z_u u + Z_w w = 0 \quad (24c)$$

where u, v, w are perturbed quantities of U, V, W from the trim condition at the starting point:

$$(U_0, V_0, W_0) = (V_T, 0, 0), \quad (\alpha_0, \beta_0) = (0, 0),$$
$$(\psi_0, \theta_0, \varphi_0) = (0, 0, 0)$$

In order to express ψ, θ in terms of α, β, φ, Eqs. (11) and (12) can be applied. Since $\psi_W(t) \equiv 0$ and $\theta_W(t) \equiv 0$ from Eqs. (23), they are reduced to

$$\tan\theta = \frac{\sin\beta\sin\varphi + \cos\beta\sin\alpha\cos\varphi}{\cos\beta\cos\alpha} \quad (25a)$$

$$\sin\psi = \cos\beta\sin\alpha\sin\varphi - \sin\beta\cos\varphi \quad (25b)$$

Substituting Eq. (25a) and Eq. (5) into Eqs. (24), and setting

$$\theta(\alpha, \beta, \varphi) = \tan^{-1}\frac{\sin\beta\sin\varphi + \cos\beta\sin\alpha\cos\varphi}{\cos\beta\cos\alpha}$$

Eqs. (24) become

$$-g\sin\{\theta(\alpha, \beta, \varphi)\} + X_u V_T(\cos\alpha\cos\beta - 1)$$
$$+ X_w V_T\cos\beta\sin\alpha + X_{\delta T}\delta T = 0 \quad (26a)$$

$$g\cos\{\theta(\alpha, \beta, \varphi)\}\sin\varphi + Y_v V_T\sin\beta = 0 \quad (26b)$$

$$g[\cos\{\theta(\alpha, \beta, \varphi)\}\cos\varphi - 1] + Z_u V_T(\cos\alpha\cos\beta - 1)$$
$$+ Z_w V_T\cos\beta\sin\alpha = 0 \quad (26c)$$

which correspond to Eqs. (17) in the previous chapter.

Since δT is contained in only Eq. (26a), the triplet (α, β, φ) can be determined by Eqs. (26b) and (26c), and then δT is obtained by substituting them into Eq. (26a). Therefore, only Eqs. (26b) and (26c) will be considered hereafter.

To prescribe the rolling maneuver or to give one more constraint as stated before, the variation of the bank angle with time, which should have suitable properties, is given by[3]

$$\varphi(t) = \frac{2\pi}{16}\left\{\cos3\pi\left(\frac{t}{T}\right) - 9\cos\pi\left(\frac{t}{T}\right) + 8\right\} \quad (27)$$

where T is the time taken for a 360-deg roll. Figure 1 shows this function for the case $T = 6$ s. If this time function is substituted into Eqs. (26b) and (26c), they become simultaneous equations with regard to $\alpha(t)$ and $\beta(t)$. Therefore, they can be solved to get $\alpha(t)$ and $\beta(t)$ as functions of time. But, because of the nonlinearity, it is not so easy to do that even by numerical method.

To do this, first Eqs. (26b) and (26c) are rewritten in the form $\beta = f(\alpha, \beta)$ and $\alpha = g(\alpha, \beta)$, where β and α in the left-hand sides are solved with respect to β-sideforce term $(Y_v v)$ and lift increment term $(Z_w w)$, respectively. Then, the

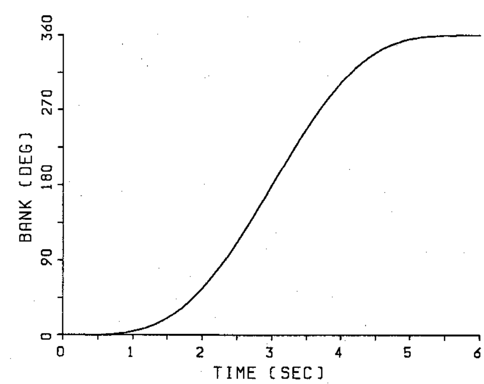

Fig. 1 Prescribed bank angle function.

successive iteration method is used to get $\alpha(t_i)$ and $\beta(t_i)$ for each moment t_i at 0.1-s intervals in the rolling maneuver. The first approximations of each iteration are taken to be $\beta_0(t_i) = -g\sin\varphi(t_i)/(Y_v V_T)$, $\alpha_0(t_i) = g(1 - \cos\varphi(t_i))/(Z_w V_T)$, which are from the simplified forms of Eqs. (26b) and (26c).

The numerical solutions of $\alpha(t)$ and $\beta(t)$ are shown as the dotted lines in Fig. 3 in the next section. The attitude angles $\psi(t)$ and $\theta(t)$ are also obtained from these solutions by Eqs. (25a) and (25b). Lastly, the moment equation [Eq. (2)] is used to get the deflections of three control surfaces. Using stability derivatives as before, Eq. (2) yields

$$L_{\delta_a}\delta a + L_{\delta_r}\delta r = \dot{P} - (I_{ZX}/I_X)(\dot{R} + PQ)$$
$$+ [(I_Z - I_Y)/I_X]QR - L_\beta\beta - L_p P - L_r R \quad (28a)$$

$$M_{\delta_e}\delta e = \dot{Q} + [(I_X - I_Z)/I_Y]RP - (I_{ZX}/I_Y)$$
$$\times(R^2 - P^2) - M_u u - M_w w - M_{\dot{w}}\dot{w} - M_q Q - M_{\delta T}\delta T \quad (28b)$$

$$N_{\delta_a}\delta a + N_{\delta_r}\delta r = \dot{R} - (I_{ZX}/I_Z)(\dot{P} - QR)$$
$$+ [(I_Y - I_X)/I_Z]PQ - N_\beta\beta - N_p P - N_r R \quad (28c)$$

All the state variables and their derivatives which appear in the right-hand sides of Eqs. (28) can be, as mentioned previously, rewritten by (α, β, φ) and their derivatives. By using $[\alpha(t), \beta(t), \varphi(t)]$ and $[\psi(t), \theta(t)]$ obtained in the prior step, and by their numerical differentiations (the central differences are used, i.e., $\dot{x}_i = (x_{i+1} - x_{i-1})/2\Delta t$), the numerical values of the right-hand sides of Eqs. (28) are all obtained. Then, Eqs. (28) become linear equations with regard to the three controls δa, δe and δr, so they can be solved easily. The attitude angles and the deflections of the three control surfaces are shown in Figs. 4 and 5 in the next section as well.

The airplane and its aerodynamic data are those of A4D for the configuration of 15,000 ft, 634 ft/s (375 kt) and are presented in Ref. 10. No limits are set to the magnitudes of the angles α, β and $\delta a, \delta e, \delta r$ in advance. The aerodynamic forces and moments used here are simple linear combinations by the stability derivatives. If more complicated but accurate aerodynamics are available, they can also be used, and there will be no problem on the calculation procedure.

Exact Solutions

Until now the assumption has been made that the three control surfaces generate only moments not forces. Although the forces generated by the control surfaces are negligible as compared with the main aerodynamic forces of each axis, i.e., lift $Z_w w$ and β-sideforce $Y_v v$, they must be taken into account in the strict sense.

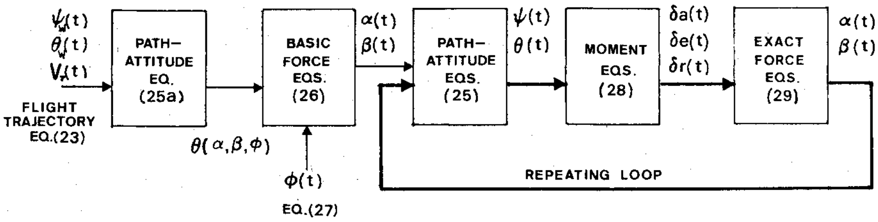

Fig. 2 Block diagram of numerical example.

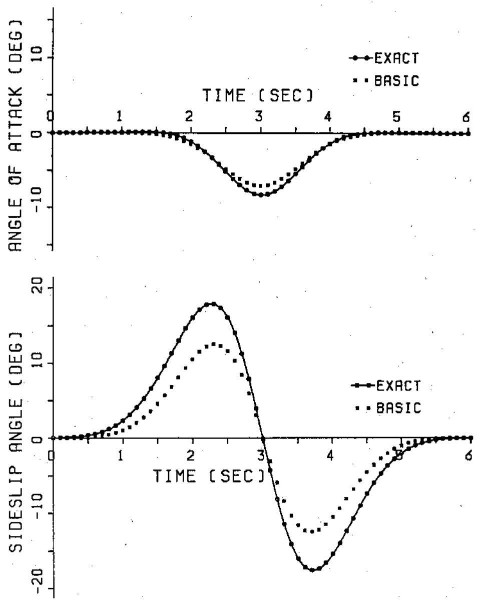

Fig. 3 Variations of angle of attack and sideslip angle.

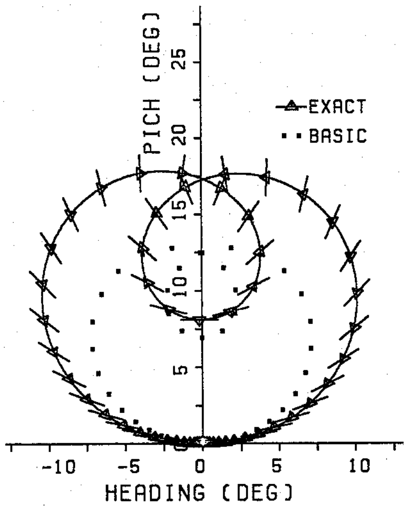

Fig. 4 Variations of attitude angles.

Now, in order to rewrite the force equations as precisely as possible, the force terms which were omitted in the preceding calculation are added. Then, the force equations [Eq. (24)] become

$$-g\sin\theta + X_u u + X_w w + X_{\delta T}\delta T + X_{\delta_e}\delta e = 0 \quad (29a)$$

$$g\cos\theta\sin\varphi + Y_v v + Y_{\delta_r}\delta r + Y_{\delta_a}\delta a = 0 \quad (29b)$$

$$g(\cos\theta\cos\varphi - 1) + Z_u u + Z_w w + Z_{\dot w}\dot w + Z_{\delta_e}\delta e = 0 \quad (29c)$$

These equations correspond to Eq. (16). Since Eqs. (29b) and (29c) include also unknown controls δa, δe and δr, they

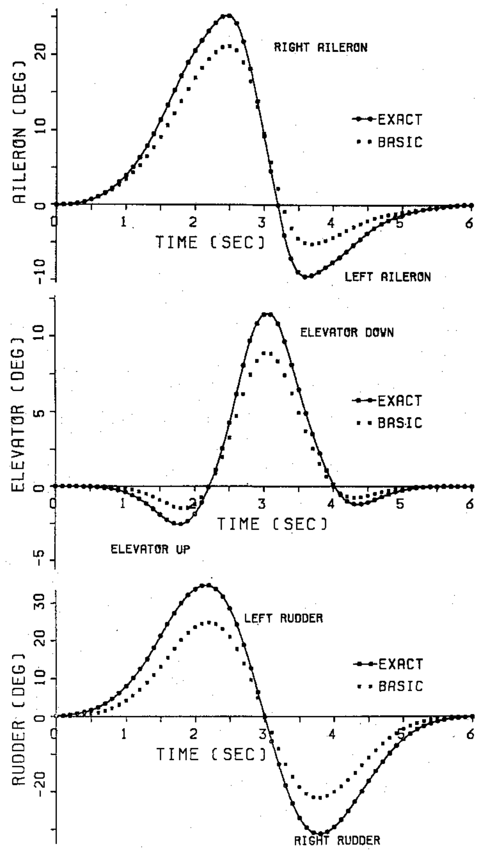

Fig. 5 Variations of control surfaces deflections.

cannot be regarded as simultaneous (differential) equations of α and β as before. However, the values of $\delta a(t)$, $\delta e(t)$ and $\delta r(t)$ have been obtained numerically in the basic solutions. Therefore, when they are substituted into Eqs. (29b) and (29c), these become simultaneous equations of $\alpha(t)$ and $\beta(t)$ again. Accordingly, $\alpha(t)$ and $\beta(t)$ can be calculated by using the successive iteration method as before. The first approximations this time are the basic solutions. Thus, the solutions $\alpha(t)$ and $\beta(t)$ obtained here are more exact than the basic solutions.

Then, following the procedures of the preceding section, $[\psi(t), \theta(t)]$ and $[\delta a(t), \delta e(t), \delta r(t)]$ can be re-obtained. These are thus more exact than the basic solutions. Furthermore, if these $\delta a(t)$, $\delta e(t)$ and $\delta r(t)$ are substituted into Eqs. (29) again and Eqs. (29) are recalculated, the solutions $[\alpha(t), \beta(t)]$ and, accordingly, $[\psi(t), \theta(t)]$ will be still more exact.

Here, it can be seen that by repeating these procedures between the force equations [Eqs. (29)] and moment equations [Eq. (28)], the solutions obtained will get more and more exact before finally converging. The exact solutions here have this sense, and they are nothing but the solutions by solving the force equations and the moment equations simultaneously by means of a numerical method. Figure 2 is a simplified block diagram describing these procedures. The exact solutions $[\alpha(t), \beta(t)], [\psi(t), \theta(t)]$ and $[\delta a(t), \delta e(t), \delta r(t)]$ are shown together with the basic solutions in Figs. 3, 4 and 5.

The numerical calculations in this section were executed in Nagoya University Computation Center.

Discussion of Numerical Result

As shown in Figs. 3, 4 and 5, the exact solutions differ considerably from the basic solutions. This is due to the assumption made when calculating the basic solutions, due to the omitted force terms in the force equations, particularly $Y_{\delta_r}\delta r$. As is seen from the conventional mechanism between the rudder and the sideslip, this force $Y_{\delta_r}\delta r$ is usually in the opposite direction to the β-sideforce $Y_v v$. Owing to this decrease of total sideforce, the magnitude of β calculated as the exact solution becomes larger than that of the basic solution. The increase of the three control surface deflections are affected by this increase of β (even elevator). It seems that it is the same with $Z_{\delta_e}\delta e$. But, because the ratio $Z_{\delta_e}\delta e/Z_w w$ is very small in comparison with $Y_{\delta_r}\delta r/Y_v v$, there is almost no influence of $Z_{\delta_e}\delta e$ on the exact solutions except slight changes of the magnitude of α and attitude angles due to that. The present flight example is a special case in that the rudder is used a great deal. However, in most flight maneuvers, especially in coordinated flights, the rudder is an auxiliary control and not used so much. Therefore, in general, the basic solutions are not as rough as this example.

It is of interest here to discuss the maneuver in the numerical example. Even though it seemed that the aileron control was the main in this maneuver, the three controls all have important parts as shown in Fig. 5. In particular, as mentioned just above, it is a distinct point of this type of maneuver that the rudder is largely used to produce the β-sideforce. Obviously, this is because the lift can not oppose gravity by itself, and the sideforce, normal to the lift, is required to fly the airplane along the straight trajectory with rolling. As the result, the conventional airplane's nose must be moved up and down and right and left during this maneuver as shown in Fig. 4. Of course, this maneuver is not coordinated. In actual aileron rolls, on the other hand, the coordination has weight rather than the straight trajectory. Therefore, as was expected, the controls of this maneuver are quite different from ordinary aileron rolls. It can be seen from the variations of the three controls in Fig. 5 that such a maneuver is very difficult for human pilots. This maneuver itself is of interest but beyond the scope of this paper.

Concluding Remarks

As is well known, to fly an airplane along a desired flight path, the lift, (sometimes) sideforce, bank and power must be controlled at each moment so that the sum of the external forces can cause the acceleration determined by the flight trajectory. Therefore, for the steering control of the airplane, the triplet (α, β, φ) are fundamental variables which are closely connected with the lift, sideforce and bank, respectively, and are controlled through the moments of the three control surfaces (indirect force control). From this simple point of view, an inverse problem of the airplane general motion has been developed.

The treatment of (α, β, φ) as key variables makes the interpretation of the complicated maneuver much easier. By the strict distinction between the attitude angles and the path angles, and because of the consideration of the degree of freedom of (α, β, φ), this approach is applicable not only to conventional maneuvers but also to new or unusual maneuvers.

The general approach has been derived from the fundamental idea of the motion and control of an airplane. The basic solutions in the numerical example were presented as an example of this general approach. However, there may be cases where the basic solutions are not enough to be exact in actual calculations. For those cases, the exact solutions based on the basic solutions were also presented.

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